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Developing a Tier 2 Multiplicative Reasoning Intervention for Third Graders

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Introduction

- Multiplicative reasoning (MR) is a key developmental understanding (Simon, 2006)
- It requires:
  - A significant conceptual shift from additive reasoning (Tzur et al., 2013).
  - A move from thinking of number as a composite unit to thinking of two composite units with transformations or coordinating operations (Steffe, 1992)
- MR as an approach to quantitative thinking is challenging for teachers to develop in their students (Carrier, 2014), yet it is foundational to advanced mathematics.

Research Design

- Developed and tested a MR intervention for third graders with mathematics disabilities
- Implementation science approach for development (see Cook & Odom, 2013)
- Employed a mix of quantitative and qualitative research methods to engage in iterative testing and revision cycles.
- An iterative testing and revision cycles, with Years 1-2 involving *Brief Learning Trial* and *Feasibility* studies to test and improve the intervention design components and Year 3 *Pilot* study will explore fidelity and the promise of the intervention for a sample of third-grade students receiving Tier 2 instruction, through a small cluster randomized control trial with students nested in classes.

Participants in Brief Learning Trials

- Two 3rd grade teachers and 10 students with mathematics difficulties (scored ≤ 12th percentile *AimswebPlus* Concepts and Applications)

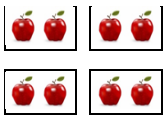
	Gender	Race/Ethnicity	EL status	Eligible for FRL
Teacher A Students	Male, *n* = 2 Female, *n* = 4	White/Not Hispanic, *n* = 3 Laotian, *n* = 1	Spanish, *n* = 3	*n* = 6
Teacher B Students	Male, *n* = 3 Female, *n* = 1	White/Not Hispanic, *n* = 3 American Indian, *n* = 1	Spanish, *n* = 2	*n* = 3
Note. None of the students was receiving special education services				

Measures

- Student Measures
  - Progress monitoring
  - Student satisfaction survey
- Teacher Measures
  - Teacher survey
  - Teacher interview
- Fidelity
  - Lesson transcriptions compared to lesson script

Sample Items from Progress Monitoring Assessment

1. Which equation represents the picture below?



   A. 2 x 4 = 8
   B. 2 + 6 = 8
   C. 4 x 2 = 8
   D. 4 + 4 = 8

2. Which division equation can be solved with the picture below?

   A. 12 ÷ 3 = 4
   B. 12 ÷ 2 = 6
   C. 12 ÷ 6 = 2
   D. 12 ÷ 4 = 3

3. Which equation is the inverse of 4 x 5 = 20?
   A. 2 x 10 = 20
   B. 5 x 4 = 20
   C. 20 ÷ 5 = 4
   D. 20 ÷ 2 = 10

4. Which of the following represents a correct fact family?
   A. 18 ÷ 2 = 9 18 ÷ 9 = 2 6 x 3 = 18 3 x 6 = 18
   B. 18 ÷ 3 = 6 6 x 3 = 18 3 x 6 = 18 18 ÷ 6 = 3
   C. 3 + 5 = 8 4 x 2 = 8 5 + 3 = 8 2 x 4 = 8
   D. 2 x 4 = 8 4 + 2 = 6 4 x 2 = 8 2 + 4 = 6

Intervention

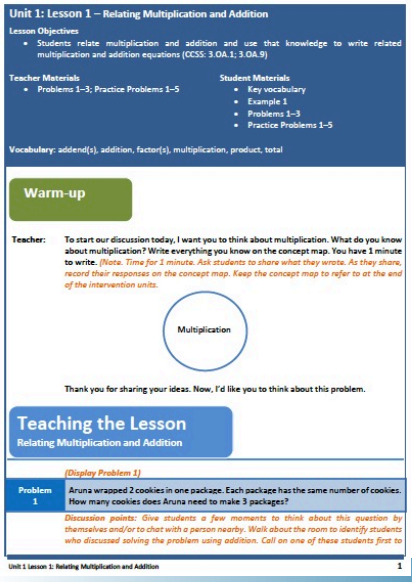
- Teachers delivered the MR intervention in small groups for 10 weeks.

Unit 1: Meaning of Multiplication	Unit 2: Strategies for Multiplication
30-minute daily instruction for 4 weeks Lessons 1-5	20-minute daily instruction for 6 weeks Lessons 1-5

Unit 1: The Meaning of Multiplication

Lesson	Objectives
1	• Relate multiplication and addition and use that knowledge to write related multiplication and addition equations. (CCSS: 3.OA.1; 3.OA.9).
2	• Relate division and multiplication and use that knowledge to write division equations. (CCSS: 3.OA.2; 3.OA.9).
3	• Relate multiplication and division using equal-size groups and understand that multiplication and division are inverse operations. Use that knowledge to write related multiplication and division equations. (CCSS: 3.OA.1, 3.OA.2, 3.OA.6)
4	• Relate multiplication and division using arrays and understand that multiplication and division are inverse operations. Use that knowledge to write related multiplication and division equations. (CCSS: 3.OA.1, 3.OA.2, 3.OA.6)
5	• Solve division problems by thinking of the corresponding multiplication. Determine the unknown whole number in a multiplication or division equation relating three whole numbers. (CCSS: 3.OA.4, 3.OA.6)

Unit 1: Sample Lesson Excerpt



Unit 1: Lesson 1 – Relating Multiplication and Addition

Lesson Objectives

- Students relate multiplication and addition and use that knowledge to write related multiplication and addition equations (CCSS: 3.OA.1; 3.OA.9)

Teacher Materials
- Problems 1–3; Practice Problems 1–5

Student Materials
- Key vocabulary
- Example 1
- Problems 1–3
- Practice Problems 1–5

Vocabulary: addend(s), addition, factor(s), multiplication, product, total

Warm-up

Teacher: To start our discussion today, I want you to think about multiplication. What do you know about multiplication? Write everything you know on the concept map. You have 1 minute to write. *(Note. Time for 1 minute. Ask students to share what they wrote. As they share, record their responses on the concept map. Keep the concept map to refer to at the end of the intervention units.*

Multiplication

Thank you for sharing your ideas. Now, I'd like you to think about this problem.

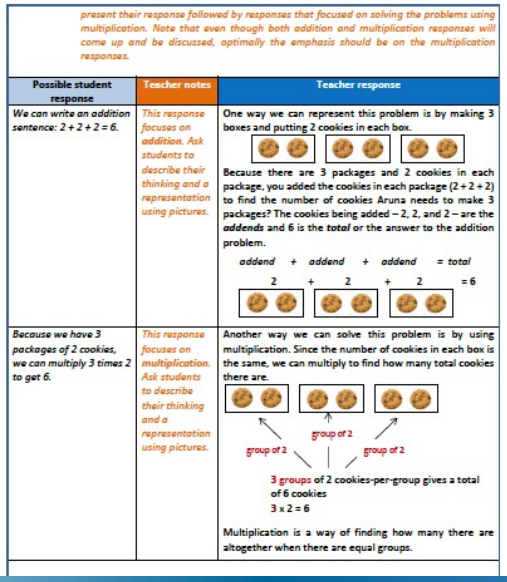
Teaching the Lesson
Relating Multiplication and Addition

(Display Problem 1)

Problem 1	Aruna wrapped 2 cookies in one package. Each package has the same number of cookies. How many cookies does Aruna need to make 3 packages?

Discussion points: Give students a few moments to think about this question by themselves and/or to chat with a person nearby. Walk about the room to identify students who discussed solving the problem using addition. Call on one of these students first to

Unit 1 Lesson 1: Relating Multiplication and Addition 1



present their response followed by responses that focused on solving the problems using multiplication. Note that even though both addition and multiplication responses will come up and be discussed, optimally the emphasis should be on the multiplication responses.

Possible student response	Teacher notes	Teacher response
We can write an addition sentence: 2 + 2 + 2 = 6.	*This response focuses on addition. Ask students to describe their thinking and a representation using pictures.*	One way we can represent this problem is by making 3 boxes and putting 2 cookies in each box. Because there are 3 packages and 2 cookies in each package, you added the cookies in each package (2 + 2 + 2) to find the number of cookies Aruna needs to make 3 packages? The cookies being added – 2, 2, and 2 – are the ***addends*** and 6 is the ***total*** or the answer to the addition problem. *addend + addend + addend = total* 2 + 2 + 2 = 6
Because we have 3 packages of 2 cookies, we can multiply 3 times 2 to get 6.	*This response focuses on multiplication. Ask students to describe their thinking and a representation using pictures.*	Another way we can solve this problem is by using multiplication. Since the number of cookies in each box is the same, we can multiply to find how many total cookies there are. group of 2 group of 2 group of 2 **3 groups** of 2 cookies-per-group gives a total of 6 cookies 3 x 2 = 6 Multiplication is a way of finding how many there are altogether when there are equal groups.

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Unit 2: Strategies for Multiplication

Lesson	Objectives
1	• Use models and equal-size groups to understand multiplication with multiples of 5 and 10. Identify patterns in the multiplication table and explain them using properties of operations. (CCSS-M: 3.OA.7, 3.OA.9)
2	• Use models and equal-size groups to understand multiplication with multiples of 2, 4, and 8. Identify patterns in the multiplication table and explain them using properties of operations. (CCSS-M: 3.OA.7, 3.OA.9).
3	• Use models and equal-size groups to understand multiplication with multiples of 3 and 6. Identify patterns in the multiplication table and explain them using properties of operations. (CCSS-M: 3.OA.7, 3.OA.9).
4	• Use models and equal-size groups to understand multiplication with multiples of 9. Apply properties of operations as strategies to multiply and divide; identify patterns in the multiplication table and explain them using properties of operations. (CCSS-M: 3.OA.5, 3.OA.7, 3.OA.9)
5	• Use models and equal-size groups to understand multiplication with multiples of 7. Apply properties of operations as strategies to multiply and divide; identify patterns in the multiplication table and explain them using properties of operations. (CCSS-M: 3.OA.5, 3.OA.7, 3.OA.9)

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Unit 2: Sample Lesson Excerpt

Unit 2: Lesson 1 – Multiplication Facts (Multiples of 5 and 10)

Lesson Objectives

- Students use models and equal-size groups to understand multiplication with multiples of 5 and 10. Students identify patterns in the multiplication table and explain them using properties of operations. (CCS-M: 3.OA.7; 3.OA.9)

Teacher Materials
- Number line diagrams; multiplication chart; multiplication fact cards
- Examples 1–5; Practice Problems 1–15

Student Materials
- Key vocabulary
- Examples 1–7
- Practice Problems 1–15

Vocabulary: Commutative property of multiplication, factors, multiplication, multiple, product

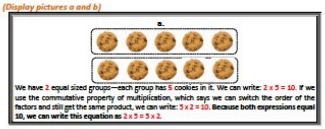
Teaching the Lesson

Multiplication Facts: 5s and 10s

Teacher: In our last lessons, we found that multiplication and division are related because they both involve equal-size groups. We're going to focus on multiplication today and think about multiplying by 5 and 10. If you think about how we wrote multiplication equations in our last lessons, what equation would you write based on these pictures? *(Display picture a and picture b and give students time to write an equation. Ask students to share their equations. After students have shared their equations, you can share the think-aloud given with each picture.)*

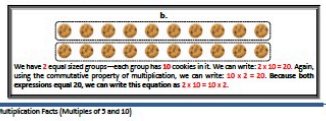
(Display pictures a and b)

a.



We have 2 equal sized groups—each group has 5 cookies in it. We can write: 2 x 5 = 10. If we use the commutative property of multiplication, which says we can switch the order of the factors and still get the same product, we can write: 5 x 2 = 10. Because both expressions equal 10, we can write this equation as 2 x 5 = 5 x 2.

b.



We have 2 equal sized groups—each group has 10 cookies in it. We can write: 2 x 10 = 20. Again, using the commutative property of multiplication, we can write: 10 x 2 = 20. Because both expressions equal 20, we can write this equation as 2 x 10 = 10 x 2.

Teacher: Let's explore other multiplication ideas with 5 and 10. *(Show a number line*

(Display Example 1a)

Example 1a Use a number line to show multiplication. Measure to a given number by a length of 5 units. Mark line segments from 0–5 to show the group of 5, count the number of line segments, and write the multiplication equation.

(Display number line)

Teacher: We can use a number line to show multiplication. Remember that we are using equal-size groups. On this number line, there will be 5 units in a group. *(Run your finger from 0–5 showing the group of 5.)* The length from 0–5 is one group. *(Mark a line segment from 0–5 using a marker.)* What multiplication equation can we write?

Students: *1 x 5 = 5.*

Teacher: *(Run your finger from 5–10, showing the group of 5. Mark a line segment from 5–10 using a marker.)* At 10, we have 2 groups of 5. What multiplication equation can we write?

Students: *2 x 5 = 10.*

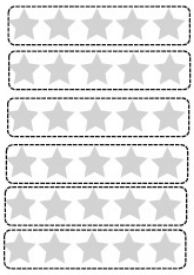
Teacher: Using your number line, mark the next group of 5 and write the multiplication equation. *(Give students time to work. Check their work and then mark it on your number line. Use a think-aloud if students were not able to do it. If they were, continue with the next instructions.)*

Unit 2: Sample Lesson Excerpt

(Display Example 4)

Example 4 Use words to describe the model below. Then write 3 multiplication equations that represent the model.

(These examples can be used as pair practice or individual work.)



Words describing the model: *There are 6 equal-size groups of stars. Each group has 5 stars in it. There are 30 stars total.*

Multiplication equations: *5 x 6 = 30 6 x 5 = 30 5 x 6 = 6 x 5*

(Point to the multiplication chart.) Now let's fill in our multiplication chart with the facts we learned today (i.e., the facts with 5 and 10 as factors). *(Note. Have students fill in the facts with 5 and 10 as factors. Model how to fill in the chart by measuring with 2s and 4s as needed. Also, point to the rows and columns and ask the question, "5 x 1 = ?" "5 x 2 = ?" and so on.)*

(Display multiplication chart)

x	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

(Pause for students to complete the chart with 5 and 10 as factors.) Let's practice finding products using the multiplication chart. Find the row (or the column) for the first factor and the column (or the row) for the second factor to figure out the product. *(Pause for students to complete.)*

5 x 2 = 10 10 x 4 = 40 5 x 9 = 45 10 x 10 = 100

x	1	2	3	4	5	6	7	8	9	10
1					5					10
2					10					20
3					15					30
4					20					40
5	5	10	15	20	25	30	35	40	45	50
6					30					60
7					35					70
8					40					80
9					45					90
10	10	20	30	40	50	60	70	80	90	100

Teacher: Good job working hard! Next, we will learn to multiply with multiples of 2, 4, and 8.

Results

- Teacher Survey: 4-point rating scale (4 = strongly agree, 1= strongly disagree)
  - Lesson scripts were helpful (*M* = 3.4)
  - Lessons provided sufficient opportunities for students to respond (*M* = 4.0)
  - Students were engaged with the instructional materials (*M* = 4.0)
  - Students would improve in multiplicative reasoning skills (*M* = 3.5)
  - MR intervention incorporated evidence-based practices (*M* = 3.9)
  - MR intervention aligned with third-grade standards (*M* = 3.6)

Results

- Teacher Interviews:
  - Students needed the repetition and re-teaching included in the lessons.
  - Student responses in the early lessons indicated they lack number and operation sense.
  - Both teachers liked the strategies and reported that their students' performance improved in terms of understanding multiplication and division as inverse operations.

Results

- Lesson Analysis
  - Logistical
    - Length of lessons too long for typical intervention class
    - Classroom management of small group instruction impacts pacing of lessons.
  - Intervention Content
    - Effective representations within the intervention: Number line and equal size groupings
    - Mathematical foundations:
      - Relationships – multiplication and addition; division and subtraction
      - Pattern identification
    - Use of precise academic language

Implications for revisions and testing

- Shorter lessons
- Increased focus on conceptual understanding
- Included more interactive components
  - Warm-up
  - Think-pair-share
  - Hands-on explorations
- Sharpened representations
- Identified classroom management and questioning strategies for inclusion in professional development

Acknowledgment

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